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Analysis of thermally optimized fin array in boiling liquids

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Abstract—Considering temperature dependent heat transfer coefficient, an optimum longitudinal fin array is investigated. The heat transfer coefficient is assumed to vary with a power-law-type formula. The heat transfer from tips of an array fins is taken into consideration. For convenience of design, two approaches in the optimization of array system are given. The system is optimized by maximizing the heat dissipation of the fin array at a fixed total fin volume. It shows that the aspect ratios of fins for an optimized array is larger than that of a single optimum fin. The results of this work are all presented in dimensionless form for the convenience of design analysis. In addition, results from this study are compared with the experimental data of previous works.

INTRODUCTION

Longitudinal rectangular fin is a common geometry for extended surfaces which are widely used in fin arrays. In many industrial applications such as aerospace, air-conditioning, electronic components, automobile radiator and heat exchangers in vessels, the weight or the available space is a primary design consideration. Thus, it is desirable to obtain the optimum design of fin arrays.

In the fundamentals of fin's heat transfer characteristics, Kern and Kraus [1] gave a thorough treatment for the optimum design of convecting fins. A comprehensive review of extended surface technology including individual optimized fin and optimization of fin array was presented by Kraus [2] for over six decades. Lately, to improve the design of extended surfaces, a lot of works [3–7] have been done in the optimization of a single fin at a specified heat duty for temperature dependent thermal parameters.

However, all the foregoing methods are mostly essentially applicable for the case of a single fin. In actual practice, an engineer rarely comes across a single fin. Besides, the recent concern over the thermal efficiency of energy conversion equipment has furthermore focused attention on the optimization of free and forced convection fin arrays. In the design of heat exchangers in vehicles and vessels, it demands that the requisite thermal load be dissipated with minimum volume and weight. This requirement can be met by choosing fin dimensions which maximize heat dissipation per fin volume. Several studies of free convection from multiple surfaces can be found in literature [8–11]. Due to the nonlinear characteristics of this problem, most of the works are investigated experimentally. In forced convection, Dhar and Arora [12] obtained the optimum fin profile and concluded that the use of individually optimum profile fins did not necessarily result in the optimum finned surfaces. The least material of convectively cooled arrays of longitudinal, rectangular fins was theoretically proposed by Bar-Cohen and Jelinek [13]. In their work, it is pointed out that the aspect ratio of the array optimum fin is found to be only marginally thinner than implied by the conventional single optimum fin. Assuming no temperature gradients in the extended surfaces, Bar-Cohen [14] theoretically predicted the optimal thickness of fin was equal to the optimal fin spacing. In addition, Bar-Cohen and Rohsenow [15] compiled a tabulation of the Nusselt number relations recommended for vertical parallel-plate fins under various thermal boundary conditions encountered in practice.

In the use of fins subject to boiling liquid, Klein and Westwater [16] performed an experiment to investigate the effect of fin's heat transfer by varying the spacing between fins. Their study was conducted mostly with copper spines. Due to the complexity of the boiling phenomenon, no optimum fin spacing was reported.

In this study, considering temperature dependent heat transfer coefficient, an optimized array with longitudinal rectangular fins is investigated. Forced convection, free convection, film boiling and nucleate boiling, which all exist in pool or flow boiling, are taken into consideration. To the best of the author's knowledge, this is the first paper to present analytically the optimum design of fin array for boiling heat transfer. For convenience of applications, two design approaches are proposed. The common input conditions are width and height of an array, base temperature, working fluid and thermal conductivity of fins. In addition to the above known values, total fin volume and fin thickness for the first model and

NOMENCLATURE

A	fin profile area [m^2]
a	dimensional constant related to a selected heat transfer mode [W m^{-2} K^{-1}]
Bi_1	dimensionless parameter, $h_b b/k$
Bi_2	dimensionless parameter, $h_b A^{0.5}/k$
b	fin thickness at fin base [m]
F	hypergeometric function
F_1	hypergeometric function, $F[1/2, m+3/2(m+1); 3/2; 1-\xi^{m+1}]$
F_2	hypergeometric function, $F[1/2, m+3/2(m+1); 3/2; \zeta]$
H	height of fin array [m]
h	heat transfer coefficient [W m^{-2} K^{-1}]
k	thermal conductivity of the fin [W m^{-1} K^{-1}]
l	fin length [m]
m	power-exponent
N	number of fins
Q	heat transfer rate of an array [W]
Q_1^*	dimensionless heat duty of an array, $Qb/(WHk\theta_b)$
Q_2^*	dimensionless heat duty of an array, $Q/(NHk\theta_b)$
s	fin spacing [m]
T	temperature of fin surface [K]
V	total fin volume [m^3]
W	width of fin array [m].

Greek symbols

α	aspect ratio of a fin, l/b
β	dimensionless geometric parameter, WHb/V
ε	ratio of fin tip to peripheral fin surface constants, a_e/a
γ	dimensionless geometric parameter, $W/(NA^{0.5})$
ψ	dimensionless temperature, θ/θ_b
θ	temperature difference between fin surface and ambient fluid, $T-T_\infty$
ξ	variable, defined in equation (3)
ζ	variable, defined in equation (4).

Subscripts, superscripts

b	fin base
e	fin tip
o	optimum
s	single fin
*	nondimensional quantity
∞	ambient.

the number of fins and fin profile area for the second one can be selected as the given conditions. The optimum aspect ratios of fins and spacing between fins in an array are then obtained and the corresponding maximum heat dissipation is evaluated. Besides, a comparison in the optimum fin spacings between the present analysis and previous experiment is made.

MATHEMATICAL ANALYSIS

Figure 1 shows the geometric definition of a longitudinal fin array with rectangular fins. The surface heat flux along the fin is considered to exhibit a power-law-type dependence on the temperature difference between the fin and the ambient fluid, i.e.

$$q = a(T-T_\infty)^m. \qquad (1)$$

The dimensional constant a and dimensionless constant m depend upon the environmental fluids and the heat transfer modes [3, 5, 17 and 18]. In the present analysis, it is assumed that there is one-dimensional steady-state heat conduction through the fins in an array, a constant thermal conductivity for fins in the array material, no heat source in the fins and base plate, a uniform temperature at the fin base, a uniform temperature for the surrounding fluid and equal fin spacing between fins. In this work, two approaches are proposed to obtain the optimum aspect ratio of fins and the interfin spaces in an array. First the optimum arrays are investigated with specified total fin volume V, width of fin array W, height of fin array H, fin thickness b, base heat transfer coefficient $h_b[= a(T_b-T_\infty)^{m-1}]$ and thermal conductivity of fin material k. Second, the array system is optimized with

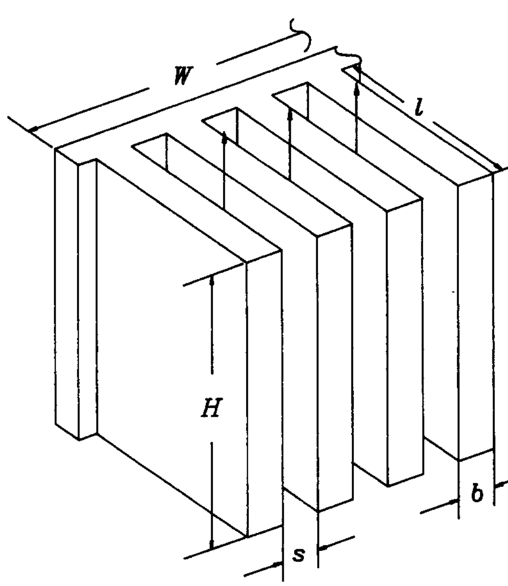

Fig. 1. Geometric definition of an array with longitudinal rectangular fins.

given W, H, h_b, k, number of fins N and fin profile area A.

Given V, W, H, b, h_b *and* k. The heat transfer rate from an array of longitudinal rectangular fins can be found by summing the thermal contribution of the interfin area and the fins [18] and is written as

$$Q = \frac{WH\theta_b}{b+s}\left\{h_b s + 2\left[\frac{kh_b b}{m+1}(1-\xi^{m+1})\right]^{1/2}\right\} \quad (2)$$

where s stands for fin spacing and

$$\xi = \psi_e(1-\zeta)^{1/(m+1)}. \quad (3)$$

In equation (3) ψ_e represents θ_e/θ_b and

$$\zeta = \frac{\varepsilon^2(m+1)}{4} \times Bi_1 \times \psi_e^{m-1} \quad (4)$$

where ε is a ratio of fin tip to peripheral fin surface constants and Bi_1 is equal to $h_b b/k$. Note that for a large number of fins, the number of interfin spaces is approximately equal to the number of fins and is expressed as $W/(b+s)$. It is desired to maximize Q by varying s. Since the total fin volume is fixed, the length of fins in an array varies accordingly. Thus, ψ_e also changes with s. In general, Q is a function of s and ψ_e. In order to maximize Q, another constraint must exist and is given as [18]

$$\left[\frac{h_b(m+1)}{kb}\times\xi^{m-1}\right]^{1/2}\times\frac{V(b+s)}{WHb} = (1-\xi^{m+1})^{1/2}F_1 - \zeta^{1/2}F_2 \quad (5)$$

where

$$F_1 = F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\xi^{m+1}\right]$$

and

$$F_2 = \left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};\zeta\right].$$

Note that equation (5) is derived from the temperature distribution of a single fin and F is a hypergeometric function which can be found in many textbooks and handbooks [19–21]. By using Lagrange's multiplier method, an optimum relationship is obtained as

$$\begin{aligned}&4\psi_e^m(2m\zeta-m-1)\xi^{-(m+3)/2}\{[Bi_1(m+1)/\\&(1-\xi^{m+1})]^{1/2}-2\}+\varepsilon^2\sqrt{Bi_1}(m+1)\\&\times(m-1)\psi_e^{m-2}\zeta^{-1/2}(1-\zeta)^{-0.5(m+3)/(m+1)}\\&\times[2(1-\xi^{m+1})^{1/2}-Bi_1\sqrt{m+1}]+4[(1-\xi^{m+1})^{1/2}\\&\times F_1-\zeta^{1/2}F_2]\Big\{2(1-\xi^{m+1})^{-1/2}\\&\times\psi_e^m(2m\zeta-m-1)+(m+1)\xi^{-1}(1-\zeta)^{1/(m+1)}\\&\times\left[1-\frac{(m-1)\zeta}{(m+1)(1-\zeta)}\right][2(1-\xi^{m+1})^{1/2}\\&-\sqrt{Bi_1(m+1)}]\Big\}=0.\end{aligned} \quad (6)$$

For any given m and Bi_1, ψ_e can be immediately evaluated from equation (6). Substituting the calculated ψ_e into equation (5) gives

$$\alpha_o = \left[\frac{\xi^{1-m}}{Bi_1(m+1)}\right]^{1/2}[(1-\xi^{m+1})^{1/2}F_1-\zeta^{1/2}F_2] \quad (7)$$

where α_o is the optimum aspect ratio, l/b, of a fin in an array.

Insulated tips of fins in an array ($\varepsilon = 0$). It is apparent that $\xi = \psi_e$ and $\zeta = 0$ is a special case of a fin array with negligible heat loss from fin tips. The design expression (6) can thus be simplified as

$$\begin{aligned}&2(1-\psi_e^{m+1})^{1/2}-[Bi_1(m+1)]^{1/2}\\&-\left\{(m-1)\left[\frac{Bi_1(1-\psi_e^{m+1})}{m+1}\right]^{1/2}\right.\\&\left.-\frac{2(m-1)(1-\psi_e^{m+1})}{m+1}+2\psi_e^{m+1}\right\}\\&\times\psi_e^{(1-m)/2}\times(1-\psi_e^{m+1})^{1/2}\\&\times F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\psi_e^{m+1}\right]=0\end{aligned} \quad (8)$$

and the optimum aspects of array fins is expressed as

$$\alpha_o = \left[\frac{\psi_e^{1-m}}{Bi_1(m+1)}\right]^{1/2}(1-\psi_e^{m+1})^{1/2} \times F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\psi_e^{m+1}\right]. \quad (9)$$

The optimum interfin spaces then becomes

$$\left(\frac{s}{b}\right)_0 = \beta\times\alpha_o - 1 \quad (10)$$

where β is equal to WHb/V.

$m = 1$: in purely forced convection, the optimum expression can be further simplified. Employing the formula [19], $F[1/2, 1; 3/2; z^2] = (1/2z)\ln(1+z/1-z)$, equation (9) is reduced as

$$\sqrt{2Bi_1}\times\alpha_o = \ln\frac{1+(1-\psi_e^2)^{1/2}}{\psi_e}. \quad (11)$$

After some mathematical manipulation and rearrangement, the dimensionless tip temperature is obtained in the form

$$\psi_e = \mathrm{sech}(\sqrt{2Bi_1}\times\alpha_o). \quad (12)$$

Substituting equation (12) into equation (8) gives

$$\sqrt{2}\times\tanh(\sqrt{2Bi_1}\times\alpha_o)$$
$$-2\sqrt{Bi_1}\times\alpha_o\times\mathrm{sech}^2(\sqrt{2Bi_1}\times\alpha_o)=\sqrt{Bi_1}. \quad (13)$$

This expression is the same as that given in the work [22].

Given W, H, N, A, h_b *and* k. The total heat transfer rate for a longitudinal fin array in this case becomes

$$Q=NH\theta_b k\left\{Bi_2\left(\frac{W}{NA^{1/2}}-\frac{1}{\sqrt{\alpha}}\right)+2\alpha^{-1/4}\left[\frac{Bi_2}{m+1}(1-\xi^{m+1})\right]^{1/2}\right\} \quad (14)$$

where ξ is given previously in equation (3). Note that the first term in the above equation gives the heat dissipation from the unfinned portion of the vertical surface and the second term gives the contribution of the fin surface. In equation (14), Bi_2 represents $h_b A^{1/2}/k$. Also, note that a large number of fins in an array is assumed, hence N equals $W/(b+s)$. In this optimization work, α is altered to maximize Q at a fixed fin profile area, A. However, ψ_e varies with α. Thus, another constraint derived from the temperature distribution of a single fin is needed and is expressible as

$$\left[(m+1)Bi_2\times\xi^{m-1}\right]^{1/2}\times\alpha^{3/4}=(1-\xi^{m+1})^{1/2}F_1-\zeta^{1/2}F_2 \quad (15)$$

where F_1 and F_2 are the same as those given in the previous design method. In this approach the fin profile area, A, of a fin is given instead of fin thickness, b. The parameter ζ is thus rewritten as $\zeta=[\varepsilon^2(m+1)/4]\times Bi_2\times\alpha^{-1/2}\psi_e^{m-1}$. The mathematical solution to this optimization problem can also be obtained by means of Lagrange's multiplier method. An optimum expression is then derived as

$$\begin{aligned}
&\psi_e^m(2m\zeta-m-1)(1-\xi^{m+1})^{-1/2}[\sqrt{Bi_2(m+1)}\\
&\times\alpha_o^{-3/4}\xi^{-(m+3)/2}-\alpha_o^{-1/2}\times\xi^{-(m+3)/2}\\
&\times(1-\xi^{m+1})^{1/2}-\zeta\psi_e^{m+1}\xi^{-(m+3)/2}(1-\xi^{m+1})^{-1/2}\\
&\times(1-\alpha_o^{-1/2})-\alpha_o^{-1/2}\zeta^{1/2}(1-\zeta)^{-0.5(m+3)/(m+1)}\\
&+\sqrt{Bi_2/(m+1)}(m-1)\psi_e\alpha_o^{1/4}\times\xi^{(m-3)/2}\\
&\times\zeta(1-\zeta)^{-m/(m+1)}+3\sqrt{Bi_2(m+1)}\alpha_o^{1/4}\xi^{(m-1)/2}]\\
&-[\sqrt{Bi_2(m+1)}\times\alpha_o^{-3/4}-\alpha_o^{-1/2}(1-\xi^{m+1})^{1/2}-\psi_e^{m+1}\\
&\times\zeta(1-\xi^{m+1})^{-1/2}]\Big\{(m-1)\psi_e^{-1}\zeta^{1/2}\\
&\times(1-\zeta)^{-0.5(m+3)/(m+1)}\\
&+\sqrt{Bi_2(m+1)}(m-1)\alpha_o^{3/4}\xi^{(m-3)/2}\\
&\times(1-\zeta)^{1/(m+1)}\times\left[1-\frac{(m-1)\zeta}{(m+1)(1-\zeta)}\right]\Big\}=0.
\end{aligned} \quad (16)$$

For any given m and Bi_2, the optimum aspect ratio of a fin in an array can be obtained by solving equation (15), replacing α with α_o and equation (16) simultaneously.

Insulated tips of fins in an array ($\varepsilon=0$). To obtain a simpler form of the design expression, an array with insulated tips of fins is investigated. In this case, $\xi=\psi_e$, $\zeta=0$ and equation (16) is reduced as

$$\begin{aligned}
&\sqrt{(m+1)Bi_2}\left(\frac{1}{\alpha_o}+3\psi_e^{m+1}\right)+[(m-1)Bi_2\times\alpha_o^{-1/4}\\
&-\alpha_o^{-3/4}]\times(1-\psi_e^{m+1})^{1/2}\\
&-(m-1)\left(\frac{Bi_2}{m+1}\right)^{1/2}(1-\psi_e^{m+1})=0.
\end{aligned} \quad (17)$$

Also, from equation (15) the optimum aspect ratio of a fin is derived as

$$\alpha_o=\left[\frac{(1-\psi_e^{m+1})^{1/2}F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\psi_e^{m+1}\right]}{\sqrt{Bi_2(m+1)}\psi_e^{(m-1)/2}}\right]^{4/3}. \quad (18)$$

Apparently, the solution procedures for this case are much easier because only one equation is needed to be solved by substituting α_o in equation (18), into equation (17). At any given m and Bi_2, ψ_e can be evaluated immediately from equation (17). The optimum aspect ratio, α_o, of a fin in an array is then obtained from equation (18) with the calculated ψ_e. In this array, the optimum fin spacing is calculated as

$$\left(\frac{s}{A^{1/2}}\right)_0=\gamma-\left(\frac{1}{\alpha_o}\right)^{1/2} \quad (19)$$

or alternatively,

$$\left(\frac{s}{b}\right)_0=\gamma\times\alpha_o^{1/2}-1 \quad (20)$$

where γ is equal to $W/(NA^{1/2})$. Note that b is a known value in equation (20) since A is a given value and α_o is evaluated.

$m=1$: employing the same formula and procedures as that described in the previous case, the dimensionless tip temperature of a fin in an array is derived as

$$\psi_e=\mathrm{sech}(\sqrt{2Bi_2}\times\alpha_o^{3/4}). \quad (21)$$

Substituting equation (21) into equation (17) gives

$$\sqrt{2Bi_2}\times\alpha_o^{3/4}[\tanh(\sqrt{2Bi_2}\times\alpha_o^{3/4}) - 3\sqrt{2Bi_2}\times\alpha_o^{3/4}\,\mathrm{sech}^2(\sqrt{2Bi_2}\times\alpha_o^{3/4})] = 2Bi_2\sqrt{\alpha_o}. \tag{22}$$

Also, this expression is identical to the recent work [23].

RESULTS AND DISCUSSION

A longitudinal rectangular fin array is optimized in two different approaches depending on the given design parameters. In this section, the array with insulated-tip fins is investigated for simplicity. The two results from different design methods are discussed, respectively, in the following.

Given V, W, H, b, h_b *and* k. From the foregoing analyses, the design procedures of this optimized array system is summarized as follows: the tip temperature of the fin in an array is first evaluated with any given Bi_1. The optimum aspect ratios of fins used in an array are then obtained. With α_o and known geometric parameter β, the optimum fin spacings between fins are subsequently solved.

Figure 2 shows the dependence of α_o, ψ_e and β_{min} on Bi_1 for $m = 0.75$, 1, 1.25 and 3. It is worthwhile to mention that the power-law exponent $m = 0.75$, 1, 1.25 and 3, represents the heat transfer modes of film boiling, forced convection, laminar free convection and nucleate boiling in applications, respectively. It is observed that α_o first decreases with Bi_1 to a minimum value then increases abruptly with increasing Bi_1. However, the tip temperature of fins decreases monotonically with Bi_1 and drops to environmental temperature at a certain maximum Bi_1. This maximum value of Bi_1 may be calculated from equation (8) and is expressed as

$$(Bi_1)_{max} = \frac{4}{m+1}. \tag{23}$$

The detailed mathematical procedures are given in Appendix 1. Apparently, there exists no optimum design for any Bi_1 greater than $(Bi_1)_{max}$. Also, from equation (8) one can see that the corresponding aspect ratio of a fin is infinity as ψ_e tends to zero for $m > 1$. For $1 > m > 0$, using equations (8) and (23), α_o is derived as

$$(\alpha_o)_{\psi_e=0} = \frac{2(1+m)}{1-m}. \tag{24}$$

Since the interfin spaces should always be positive, there exists a minimum value of β as shown in the bottom of Fig. 2. It is apparent that β_{min} is equal to $1/\alpha_o$ from equation (10). No optimum fin spacing of an array is found for any given β less than β_{min}.

It is interesting to learn that there exist two identical α_os at two different Bi_1s for all heat transfer modes on the top of Fig. 2. In applications, it is suggested that a smaller Bi_1 be selected. Because the tip temperature of a fin decreases with Bi_1, the fin efficiency is lower for a larger Bi_1. The phenomenon of two α_o is further illustrated in Fig. 3 for $m = 1.25$ and 3. It shows that the maximum heat transfer rates of arrays only increase with Bi_1. Also, note that the difference in Q_1^* is not so pronounced at differing α for Bi_1 close to $(Bi_1)_{max}$. This is because fins work inefficiently at a larger Bi_1. The cause of maximum heat transfer rate of an array can be more clearly illustrated by showing the heat dissipations from finned surfaces and interfin surfaces, respectively. Figure 4 displays these results for $Bi_1 = 0.01$, 0.8, 1.77 for $m = 1.25$ and $Bi_1 = 0.01$ for $m = 3$. Since the total fin volume and fin thickness are fixed, the fin length and fin spacings increase with

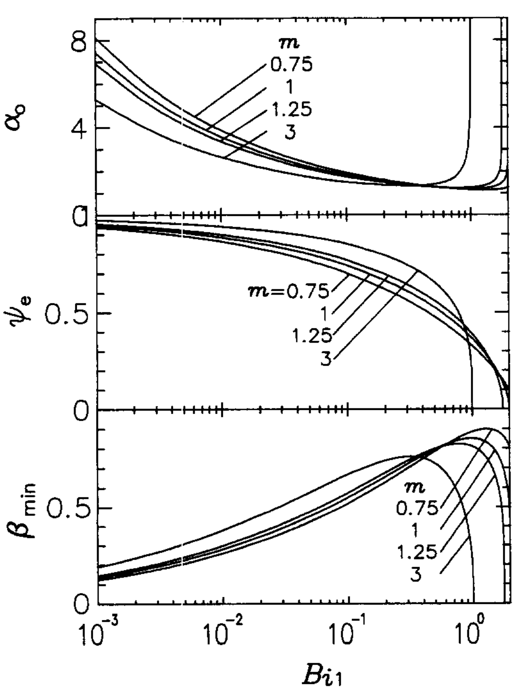

Fig. 2. Dependence of α_o, ψ_e and β_{min} on Bi_1 for $m = 0.75$, 1, 1.25 and $3(\varepsilon = 0)$.

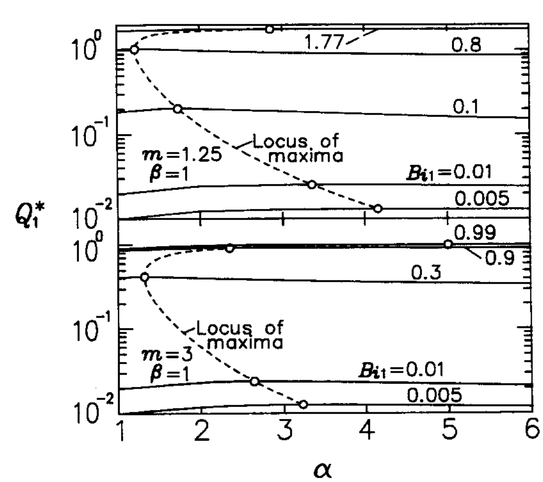

Fig. 3. Dependence of Q_1^* on α for $\beta = 1$ and $m = 1.25$ and $3(\varepsilon = 0)$.

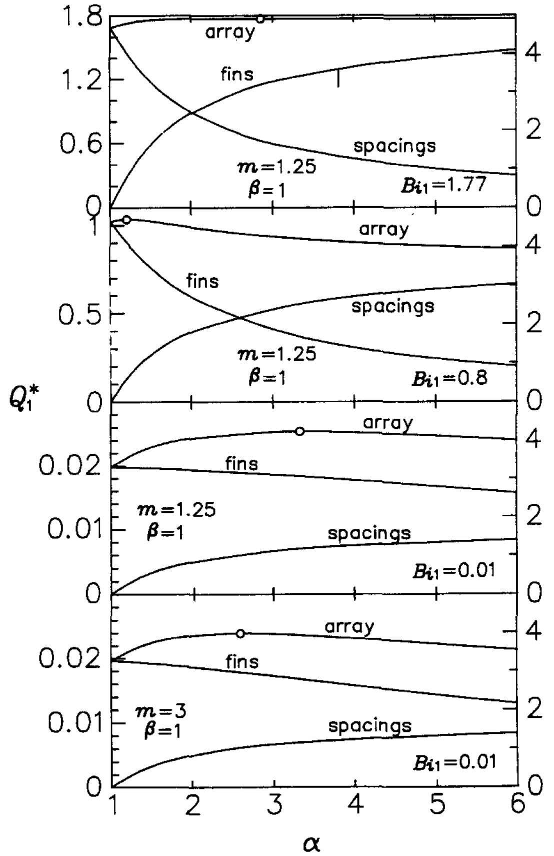

Fig. 4. The dimensionless heat flow dissipated by fins, spacings and an array at differing α for $\beta = 1$ and $m = 1.25$ and $3(\varepsilon = 0)$.

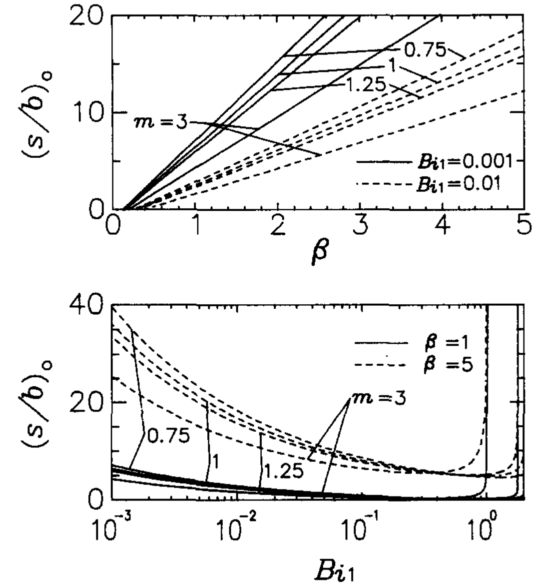

Fig. 5. Influence of β and Bi_1 on $(s/b)_0$ for $\varepsilon = 0$.

the aspect ratios of an array fin. Thus, with fixed base temperature, as α increases, the heat dissipated from fin spaces increases accordingly. However, the heat dissipations from fins tend to reduce because large numbers of smaller fins have turned into small numbers of larger fins. Hence, with the conflicting trends in heat dissipations from fins and spacings, a maximum heat transfer rate may occur at a certain α which is designated as α_o. For $m = 1.25$, it is observed that the total heat transfer rate of an array are dominated by fins' heat transfer at a smaller Bi_1. In addition, the heat dissipation from fins decreases slowly with α as can be seen from the bottom two figures in Fig. 4. With an increase of Bi_1, the heat duty of fins tends to decrease very fast whereas the heat dissipation of interfin area increases enormously on increasing α. This is due to the fact that the tip temperatures of array fins tend to zero at a larger α and the heat transfer rates from fins reduce. Thus, the maximum heat transfer rate of an array occurs at a smaller α_o. Upon increasing Bi_1 close to $(Bi_1)_{max}$, a slight variation of heat flow from these inefficient fins is observed whereas a significant increase in heat dissipation from fin spaces is attained. The optimum aspect ratios of the fins in an array therefore becomes larger. At the same Bi_1, the dimensionless heat transfer rates from fin spacings are almost identical for $m = 1.25$ and 3. However, the heat duty from fins drops more quickly on increasing α for $m = 3$ than for $m = 1.25$. Thus, the maximum heat transfer occurs at a smaller α_o for $m = 3$.

Figure 5 depicts the dependence of $(s/b)_0$ on β and Bi_1. It is shown that the optimum fin spacing between fins is large at a smaller Bi_1 due to a smaller fin thickness. At a specified geometric parameter β, $(s/b)_0$ decreases with m. Note that $(s/b)_0$ first decreases with Bi_1 to a minimum value then increases abruptly with increasing Bi_1. Similar trends are observed for different βs. Because of the action of buoyancy forces, the fluid of higher temperature travels upwards along each vertical surface of the array. Therefore, the boundary layers grow thicker as the fluid proceeds upward. The boundary layer thicknesses on the fins increase with the fin height H. Also note that the fin spacings must be twice the boundary layer thickness at the top of isothermal fins to avoid boundary layer interference. Hence, the larger the geometric parameter is, the larger the optimum interfin space is obtained. In addition, no optimum fin spacing is found for $Bi_1 > (Bi_1)_{max}$. The dependence of Nb/W on Bi_1 is given in Fig. 6. With the known values of b and W, the optimum number of fins in an array can be readily obtained. It shows that Nb/W first increases with Bi_1 to a maximum value then decreases with increasing Bi_1. Furthermore, Nb/W increases with β at a given Bi_1. This can be comprehended that the optimum number of fins increase as $(s/b)_0$ decreases at a fixed W.

Given W, H, N, A, h_b *and* k. In this method, the tip temperature of fins in an array are first solved at a given Bi_2. The optimum aspect ratios of fins are then evaluated. Subsequently, $(s/A^{0.5})_0$ is obtained with a specified γ. Figure 7 depicts the dependence of α_o, ψ_e

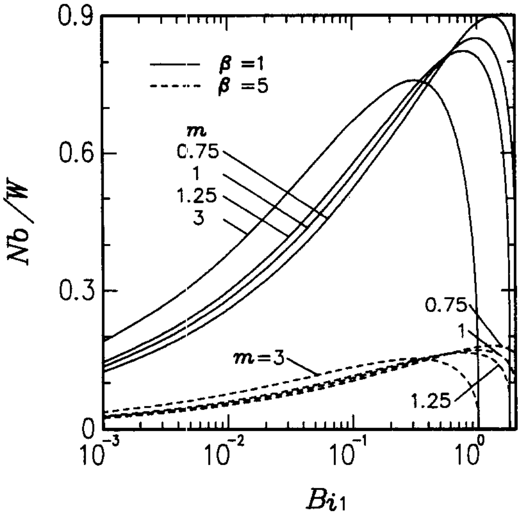

Fig. 6. Dependence of Nb/W on Bi_1 for $\beta = 1$ and 5 and $m = 0.75$, 1, 1.25 and 3($\varepsilon = 0$).

and $\gamma_{\min}$ on Bi_2. As the heat dissipations from the fin spacings vanish, equation (17) becomes

$$[(m-1)-2(2m+1)\psi_e^{m+1}] \times F\left[1, \frac{m}{m+1}; \frac{3}{2}; 1-\psi_e^{m+1}\right] + m + 1 = 0. \quad (25)$$

The above equation is the same as that of a single optimum longitudinal rectangular fin [24]. It is also observed that a minimum α_o exists at a certain Bi_2. However, the tip temperatures of fins decrease monotonically with Bi_2. For $0 < m < 1$, a maximum value of Bi_2 is found at $\psi_e = 0$ and is derived as

$$(Bi_2)_{\max} = [(1-m)(1+m)]^{-1/2}. \quad (26)$$

At $(Bi_2)_{\max}$, the optimum aspect ratios of fins becomes

$$(\alpha_o)_{\psi_e=0} = \frac{1+m}{1-m}. \quad (27)$$

The complete details of the algorithm are given in Appendix 2. Also, no optimum design is found for $Bi_2 > (Bi_2)_{\max}$. For $m \geq 1$, the zero dimensionless tip temperature occurs at infinite Bi_2. It is indicated that there always exists an optimum fin array for a given finite Bi_2 in the case of $m \geq 1$. In addition, there exists a minimum value of γ for positive spacings between fins. This is given in the bottom of Fig. 7. The influences of m and γ on dimensionless interfin spaces $(s/A^{0.5})_0$ for $\gamma = 1$ and 3 are shown in Fig. 8. Note that $(s/b)_0$ is also given for reference. This is due to the fact that b is a known value since α_o is calculated in advance for a specified fin profile area. It is observed that $(s/b)_0$ and $(s/A^{0.5})_0$ both first decrease with Bi_2 to a minimum value then increase with Bi_2.

The dependence of Q_2^* on α is displayed in Fig. 9 for $\gamma = 1$ and $m = 1.25$ and 3. In these figures, the variation in Q_2^* is obvious at smaller values of α, but no significant difference in Q_2^* is shown at larger αs, however, a maximum Q_2^* can still be detected. This can be explained that Q_2^* is merely the summation of the heat transfer from a fin and a spacing whereas Q_1^* is contributed to all the fins and inter-fin spaces of an array. The effect diminishes hence the difference between maximum Q_2^* and the values nearby is not so pronounced. The cause of maximum Q_2^* on differing α for $m = 1.25$ and 3 are given in Fig. 10. The dimensionless space in a fin array can be obtained directly from equation (20). It is apparent that $s/A^{0.5}$ increases parabolically with α and the heat transfer from fin spacings thus also increases with α. Aside from the maximum heat dissipation of fin array, it is worthy of noting that a maximum heat transfer rate of a fin is

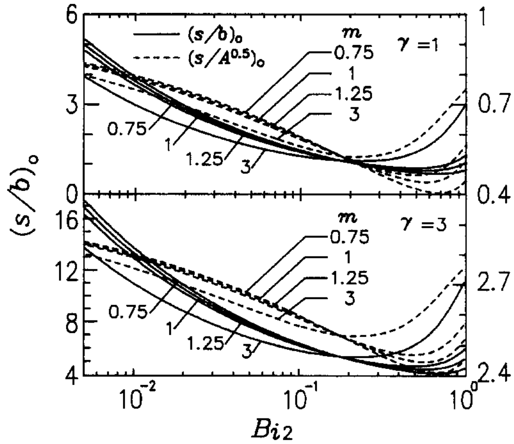

Fig. 8. Dependence of $(s/b)_0$ or $(s/A^{0.5})_0$ on Bi_2 for $\gamma = 1$ and 3 and $m = 0.75$, 1, 1.25 and 3($\varepsilon = 0$).

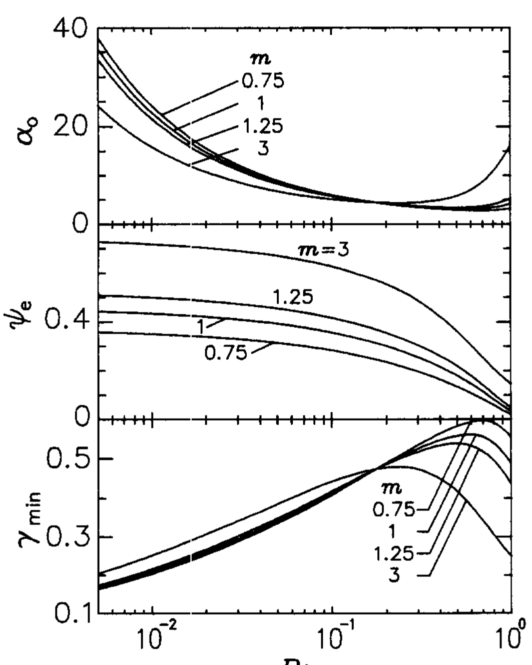

Fig. 7. Dependence of α_o, ψ_e and $\gamma_{\min}$ on Bi_2 for $m = 0.75$, 1, 1.25 and 3($\varepsilon = 0$).

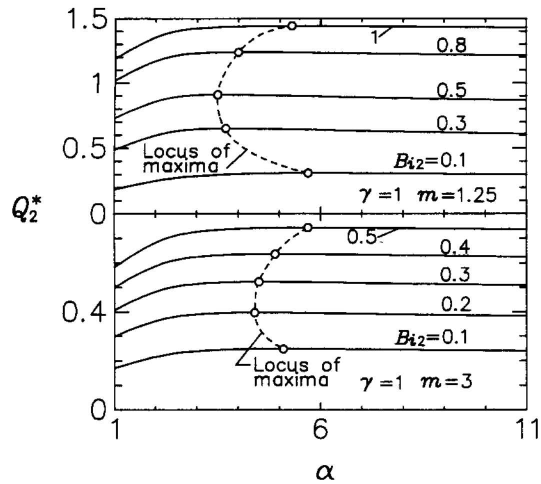

Fig. 9. Effects of Bi_2 on Q_2^* for $\gamma = 1$ and $m = 1.25$ and $3(\varepsilon = 0)$.

also observed at a certain α for $Bi_2 = 0.1$ and 0.5. From [24], the optimum aspect ratio of a rectangular fin can also be expressed as a function of Bi_2 and is written as $(\alpha_o)_s = 0.939Bi_2^{-2/3}$ for $m = 1.25$ and $(\alpha_o)_s = 0.653Bi_2^{-2/3}$ for $m = 3$. Note that $(\alpha_o)_s$ is equal to 0.939 at $Bi_2 = 1$ for $m = 1.25$. Thus, the optimum aspect ratio of a single fin does not show on the top of Fig. 10. A significant improvement in the heat transfer

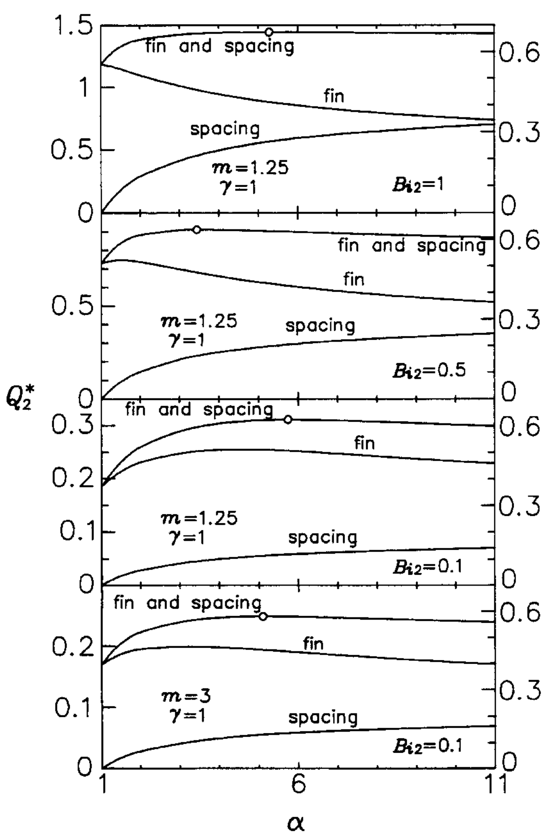

Fig. 10. The dimensionless heat flow dissipated by a fin, a fin spacing and a summation of both at differing α for $\gamma = 1$ and $m = 1.25$ and $3(\varepsilon = 0)$.

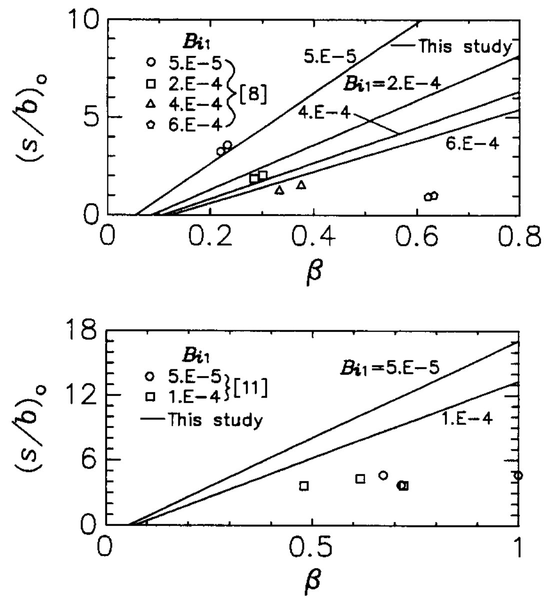

Fig. 11. Comparison of $(s/b)_0$ between the predictions of present model with insulated fin tips for $m = 1.33$ and experimental data [8, 11].

from interfin space is attained by increasing Bi_2. This is owing to the fact that the augmented base heat transfer coefficient will result in an increase in the heat transfer rate at wall, $h_b\theta_b Hs$. In view of the bottom two figures of Fig. 10, a very slight difference of the heat dissipation from fin spacing is shown between $m = 1.25$ and 3 for $Bi_2 = 0.1$. Also, the heat dissipation from spacing is smaller compared with fin's heat transfer especially at a smaller Bi_2. Thus, the heat transfer characteristics of a fin are dominant factors in the study of an optimized fin array in the present case. Consequently, the optimum aspect ratio of the array fin is larger for $m = 1.25$ than that for $m = 3$ because $(\alpha_o)_s$ is larger for $m = 1.25$ than for $m = 3$.

Regarding thermally optimal spacing of fin arrays in free convective heat transfer, a few experiments were conducted. It is found that all the data are subject to turbulent natural convection because all the Rayleigh numbers are greater than 10^9. Figure 11 shows the comparison in $(s/b)_0$ between the results predicted by the proposed model for $m = 1.33$ and experimental data. In the upper figure, it is noted that the present analysis compares favorably with the work [8] only at smaller values of Bi_1. This is due to the fact that a larger Bi_1 represents shorter and fatter fins used in an array. The proposed one-dimensional model thus fails to predict $(s/b)_0$ well. To see the difference between non-isothermal fins and isothermal fins used in an array, a comparison of this study and previous work [11] is given in the lower part of Fig. 11. It should be pointed out that the data are obtained at an experiment by controlling the temperature of whole fins to be uniform. As can be seen $(s/b)_0$ is smaller for an isothermal array fins. Also, it is worthwhile to point out that Bar-Cohen [14] theoretically predicted $(s/b)_0$

to be unity for isothermal fins in an array. This value is even smaller. To validate the present models, more data especially from boiling experiment are needed.

CONCLUDING REMARKS

In this work, considering temperature dependent heat transfer coefficient, design methods of an optimum array with longitudinal rectangular fins are presented. At a given working fluid, base temperature, thermal conductivity of fins and fin thickness (or fin profile area), the optimum aspect ratios of fins in an array are first solved. The fin spacings of the optimized array are then evaluated with the input of geometric parameter β (or γ). It is found that the aspect ratios of fins as well as interfin spaces in the optimized fin arrays first decrease with Bi_1 (or Bi_2) to minimum values then increase with Bi_1 (or Bi_2). For the model of given fixed total fin volumes, an increase in α will decrease the heat flow dissipated by fins but will increase heat transfer from fin spacings. Because of the conflicting trends, an optimum array corresponding to a maximum heat dissipation exists. In the model of the specified number of fins, a maximum heat transfer rate of a fin first occurs at a certain α while increasing α. A further increase in α will reduce the fin's heat transfer, however, the heat dissipated by interfin area will increase. Hence, an array with maximum heat dissipation also exists. It is shown that α_o of an array fin is larger than that of a single fin.

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APPENDIX 1

For $m < 1$, as ψ_e tends to zero, equation (8) may be reduced as

$$\sqrt{Bi_1(m+1)} = 2. \tag{A1}$$

For $m \geqslant 1$, multiplying equation (8) by $\psi_e^{(m-1)/2}$, one obtains

$$\begin{aligned} &2\psi_e^{(m-1)/2}(1-\psi_e^{m+1})^{1/2} \\ &-\psi_e^{(m-1)/2}[B_1(m+1)]^{1/2} \\ &-\left\{(m-1)\left[\frac{Bi_1(1-\psi_e^{m+1})}{m+1}\right]^{1/2}\right. \\ &\left.-\frac{2(m-1)(1-\psi_e^{m+1})}{m+1}+2\psi_e^{m+1}\right\} \\ &\times(1-\psi_e^{m+1})^{1/2}\times F\left[\frac{1}{2},\frac{m+3}{2(m+1)};\frac{3}{2};1-\psi_e^{m+1}\right]=0. \end{aligned} \tag{A2}$$

In the case of $m = 1$, an expression identical to equation (A1) can be immediately obtained from the above equation for $\psi_e = 0$. For $m > 1$, employing the formula [21], $F[a, b; c; 1] = \Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\times\Gamma(c-b)]$ gives the same result as that given in equation (A1) as ψ_e approaches zero.

APPENDIX 2

For $1 > m > 0$, substituting equation (18) into equation (17), using the formula [21], $F[a,b;c;x] = (1-x)^{c-a-b}F[c-a,c-b;c;x]$ and after rearrangement gives

$$Bi_2^{2/3} = \frac{(1-\psi_e^{m+1})^{0.5}[1-(1-\psi_e^{m+1})\dfrac{1-m}{1+m} \times \zeta - 3\psi_e^{m+1}\zeta]}{(1+m)^{-1/3}\zeta^{2/3}(1-\psi_e^{m+1})^{1/3} \times [(1+m)(1-\psi_e^{m+1})^{-0.5}\zeta^{-1}-1+m]} \quad \text{(A3)}$$

where ζ is denoted as $F[1, m/(1+m); 3/2; 1-\psi_e^{m+1}]$. Incorporating the formula [21], $(d/dx)\,F[a,b;c;x] = (ab/c)\times F[a+1,b+1;c+1;x]$ and applying l'Hôpital's rule yield

$$\lim_{\psi_e \to 0} Bi_2^{2/3} = [(1-m)(1+m)]^{-1/3}. \quad \text{(A4)}$$

Rearranging the above equation gives the result of $(Bi_2)_{\max}$ as expressed in equation (26). With the help of equation (18) and after some transformations, the aspect ratios of optimum array fins are obtained as

$$\alpha_0^{3/4} = \left[\frac{1}{(1+m)(Bi_2)_{\max}}\right]^{0.5} F\left[1, \frac{m}{1+m}; \frac{3}{2}; 1\right] \quad \text{(A5)}$$

for $\psi_e = 0$. Substituting equation (A4) into equation (A5) and using the formula described in Appendix 1 give the expression as shown in equation (27).